Vibration Analysis of Structures by Component Mode Substitution

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A method is presented to determine the vibration modes of a complex structural system by using component vibration modes. The structural system is considered to be an assemblage of subsystems or components. The vibration modes for each component are determined separately and then used to synthesize the system modes. The number of component modes used may be truncated to reduce the number of generalized coordinates required for a vibration analysis. Only component vibration modes are retained as generalized coordinates when the system modes are obtained; hence, the method is particularly suitable for structures with a large number of component interface coordinates, such as finite-element shell models. The boundary conditions used for determining component vibration modes can be either free-free or constrained. An optional technique to modify the component modes is included in order to obtain more accurate system modes. Numerical results from two examples are included.

Nomenclature

K = system modal stiffness matrix using free-free component modes
K^c = system modal stiffness matrix using constrained component branch modes
k = component stiffness matrix
$\bar{k}$ = reduced component stiffness matrix
k^c = system stiffness matrix using branch components
M = system modal mass matrix using free-free component modes
M^c = system modal mass matrix using constrained component branch modes
m = component mass matrix
$\bar{m}$ = reduced component mass matrix
T_c = component constrained motion transformation or component constraint modes
T_L = component coupling transformation matrix
T_n = transformation matrix using free-free component modes. The subscript n denotes various operations
T_n^c = transformation matrix using constrained component branch modes. The subscript n denotes various operations
T_r = component reducing transformation matrix
ϕ = matrix of free-free component modes
ϕ^c = matrix of constrained component branch modes
ϕ_D^c = system component mode matrix using constrained branch modes
ϕ_m^c = system modal mode matrix using contrained branch modes
ϕ_D = system component mode matrix using free-free component modes
ϕ_m = system modal mode matrix using free-free component modes
p = vector of generalized forces of a component
q = vector of generalized displacements of a component
$\hat{q}_c$ = vector of constraint-mode displacements at the noninterface coordinates
$\hat{q}_n$ = vector of fixed-constraint-mode displacements at the noninterface coordinates
ξ = vector of generalized modal coordinates for a free-free component
ξ^c = vector of generalized modal coordinates for a constrained branch component
$\xi_m{}^c$ = vector of system generalized modal mode coordinates using constrained branch component modes
ξ_m = vector of system generalized modal mode coordinates using free-free component modes
(KE) = kinetic energy
(KE_c) = kinetic energy using constrained motion
(KE_T) = total kinetic energy of uncoupled components
(PE) = potential energy
(PE_c) = potential energy using constrained motion
(PE_T) = total potential energy of uncoupled components
β_i = generalized mass of ith component mode
ω_i = circular frequency of ith component mode

Subscripts

a = component a
b = component b
ab = system (components a and b coupled together)

Superscripts

a = component a with interface loading
b = component b with interface loading
T = transposed matrix
— = interface coordinates
^ = noninterface coordinates
. = first time derivative
.. = second time derivative

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Introduction

COMPONENT mode substitution methods are useful for reducing the number of generalized coordinates required in a vibration analysis. The complete structural system is assumed to be composed of a group of subsystems or components that are interconnected. The vibration modes for each component are determined separately and then mathematically combined to synthesize the system modes. Either free-free or constrained boundary conditions may be used to determine the component modes. A primary advantage of component mode substitution is the ability to select fewer generalized coordinates and still obtain satisfactory results.

The original applications were to statically-determinate beam-type models[1,2] and have been extended to complex redundant models.[3–9] Gladwell[2] proposed a component modal substitution method that is suitable for statically determinate models. Using this method, the component modes are determined by allowing the component to vibrate with distortion, while all other components are assumed attached and rigid. This procedure produces rigid-body inertia loadings on the component due to the connected components.

Hurty[3–5] developed a comprehensive method of component modal substitution that is applicable to redundant complex structures. Using this method, the natural modes of vibration for each component are determined by fixing all interface coordinates of the component under consideration. The motion of each component is then written with reference to rigid-body modes, redundant constraint modes, and fixed-constraint natural vibration modes.

Bajan and Feng[6] and Craig and Bampton[7] used a method similar to Hurty's. The natural modes of vibration for each component are determined with all component interface coordinates fixed. The motion of each component is then written with reference to the constraint modes and fixed-constraint natural vibration modes.

Goldman[8] presented a method that used free-free component vibration modes. The connection problem involves component rigid-body and elastic vibration modes. This obviates a constraint mode analysis, but as the author stated, the errors introduced under certain cases can be quite large.

Hou[9] presented another method using free-free component modes. The connection problem requires a specific selection of as many component modes as there are connections. The success of this method depends on the selection of component modes used for the connection transformation. An improper selection of component modes for the connection may result in ill-conditioned transformation matrices.

The method presented here uses either constrained component modes, or free-free component modes, generally defined as the component modes. It does not require that the generalized coordinates of the static constraint modes appear in the final analysis of the system modes. This is particularly important for structures having a large number of interface coordinates, such as finite-element shell models. When there are many interface coordinates, the allowable size limitations of eigenvalue-eigenvector computer programs may be exceeded if constraint modes must be carried along as generalized coordinates.

In this method, connections are handled by using component stiffness matrices. Static constraint modes are used when component modes contain a fixed interface. However, these constraint modes are eliminated by the use of "branch components." A branch component is a component whose motion will be defined relative to interfaces. In the general concept, the selection of boundary conditions in obtaining component modes is arbitrary. That is, either free-free or constrained boundary conditions may be selected, except for one condition: if an interface of a component is fixed, then the corresponding interface of the connected component must be free. Any component containing a fixed interface is referred to as a constrained "branch component." This allows the constraint modes to be used for the connection, but also allows generalized coordinates of the constraint modes to be eliminated. In contrast, the method suggested by Hurty requires that all interface coordinates be fixed for the determination of component modes. Because of this, the static constraint modes must be carried along as generalized coordinates.

The accuracy of this method can be improved by applying two types of external loading to the free interface coordinates of the component under consideration. The first type of loading is a stiffness loading, which represents the reduced stiffness properties of the system, and the second is an inertial loading, which represents the reduced mass properties of the system. Both the reduced-stiffness and the reduced-mass properties are formed by a reduction process in which the stiffness and mass matrices of the system are reduced to the interface coordinates of the component being considered. This accuracy-improvement technique enables the component modes to be modified by including approximate dynamic effects of the system, and produces component modes that resemble the system modes. The technique is based on a Rayleigh-Ritz concept of component mode substitution in which the system modes are considered to be formed as linear combinations of the component modes. Therefore, it is expected that even better results will be obtained if the component modes closely resemble the system modes. The results obtained using this technique have been extremely accurate.

This paper consists of four sections. The first section describes the reduced-mass and stiffness matrices, shows their derivation, and presents basic assumptions. The second section presents the analysis of component modes subject to interface loading. The third section describes the component mode substitution method using constrained component branch modes, and the fourth section describes the component mode substitution method using free-free component modes. Two numerical examples are presented.

Reduced Stiffness and Mass Properties

Reduced-mass and stiffness matrices[10] are used to simulate attached components for interface-loaded component modes. To reduce a component mass or stiffness matrix, the motions of the component's interior coordinates $\hat{q}$ must be constrainted to move only with respect to the motion of the interface coordinates $\bar{q}$. The constrained-motion relationship is referred to as a reducing coordinate transformation. A convenient way to determine the transformation is by using a static reduction process. As an example, consider Component b in Fig. 1.

For Component b, the linear relationship between the externally applied forces and the displacements may be written, using a stiffness matrix, as

$$p_b = k_b q_b \tag{1}$$

Let the coordinates of Component b be divided into two sets: one set consists of the coordinates located at the interface connecting with Component a, and the other consists of those coordinates not located at the interface. Eq. (1) can be partitioned as

$$\begin{Bmatrix} \bar{p} \\ \hat{p} \end{Bmatrix}_b = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_b \begin{Bmatrix} \bar{q} \\ \hat{q} \end{Bmatrix}_b \tag{2}$$

The motion transformation describes the interior displacements of a component due to the displacements of its interface coordinates. Thus, no applied forces are acting at the noninterface coordinates,

$$\hat{p}_b = 0 \tag{3}$$

Equations (2) and (3) can then be used to show that the relationship of the constrained noninterface motion of Component b is

$$\hat{q}_b \text{ (for } \hat{p}_b = 0) = \hat{q}_{c_b} = T_{c_b}\bar{q}_b \tag{4}$$

where

$$T_{c_b} = -k_{22_b}{}^{-1}k_{21_b} \tag{5}$$

From Eq. (4) the total constrained motion of Component b can be written as

$$q_b \text{ (for } \hat{p}_b = 0) = \begin{Bmatrix} \bar{q} \\ \hat{q} \end{Bmatrix}_b = T_{r_b}\bar{q}_b \tag{6}$$

where

$$T_{r_b} = \begin{bmatrix} I \\ T_{c_b} \end{bmatrix} \tag{7}$$

The reduced stiffness matrix is formed from the potential energy of Component b. This potential energy may be written in quadratic form as

$$(PE)_b = \tfrac{1}{2} q_b^T k_b q_b \tag{8}$$

The potential energy of constrained Component b, $(PE_c)_b$, is then formed by substituting Eq. (6) into Eq. (8)

$$(PE_c)_b = \tfrac{1}{2} \bar{q}_b^T \bar{k}_b \bar{q}_b, \tag{9}$$

where

$$\bar{k}_b = T_{rb}^T k_b T_{rb} \tag{10}$$

and where Eq (10) defines the stiffness matrix of Component b only in terms of interface coordinates.

The reduced mass matrix for Component b can also be similarly formed from the kinetic energy of Component b. The kinetic energy of Component b may be written in quadratic form as

$$(KE)_b = \tfrac{1}{2} \dot{q}_b^T m_b \dot{q}_b \tag{11}$$

By substituting the time derivative of Eq. (6) into Eq. (11), we obtain the kinetic energy of constrained Component b as

$$(KE_c)_b = \tfrac{1}{2} \dot{\bar{q}}_b^T \bar{m}_b \dot{\bar{q}}_b \tag{12}$$

where

$$\bar{m}_b = T_{rb}^T m_b T_{rb} \tag{13}$$

and where Eq. (13) defines the mass matrix of Component b only in terms of interface coordinates.

Component Modes Using Interface Stiffness and Inertia Loading

Component modes that include mass and stiffness interface loadings are used to improve the accuracy of the results obtained for the system. The previously developed reduced-mass and stiffness matrices are indicated as they appear.

The total potential energy for uncoupled Components a and b can be written in quadratic form as

$$(PE_T) = \tfrac{1}{2} q_a^T k_a q_a + \tfrac{1}{2} q_b^T k_b q_b \tag{14}$$

Components a and b are coupled together by constraining the interface coordinates of Component b so that they are identical to the corresponding interface coordinates of Component a. This constraint is expressed as

$$\bar{q}_b = \bar{q}_a \tag{15}$$

Equation (15) assumes that the same reference coordinate system is used for both components. Different coordinate reference systems will require a rotational coordinate transformation using direction cosines Therefore, the coordinate transformation that couples Components a and b together becomes

$$\bar{q}_b = T_L q_a = [I \ \ O] \begin{Bmatrix} \bar{q} \\ \hat{q} \end{Bmatrix}_a \tag{16}$$

The reducing transformation, Eq. (6), and the coupling transformation, Eq. (16), can be substituted into Eq. (14) to obtain the total potential energy of the constrained system, $(PE)^a$:

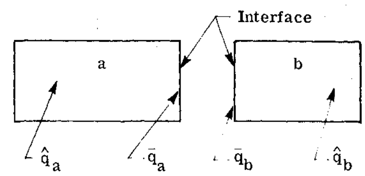
$$(PE)^a = \tfrac{1}{2} q_a^T k^a q_a \tag{17}$$

where

$$k^a = k_a + T_L^T \bar{k}_b T_L \tag{18}$$

and where the reduced stiffness matrix $\bar{k}_b$, previously defined by Eq. (10), appears in Eq. (18).

Fig. 1 Components *a* and *b*.



The kinetic energy of the constrained system, $(KE)^a$, is formed in a similar manner by taking the time derivative of Eq. (6) and (16):

$$(KE)^a = \tfrac{1}{2} \dot{q}_a^T m^a \dot{q}_a \tag{19}$$

where

$$m^a = m_a + T_L^T \bar{m}_b T_L \tag{20}$$

and where the reduced mass matrix $\bar{m}_b$, previously defined by Eq. (13), appears in Eq. (20).

The homogeneous equations of motion for the undamped constrained system can be easily formed from the potential and kinetic energy expressions either by inspection of Eqs. (17) and (19), or by substituting the potential and kinetic energies into Lagrange's equation. This will result in

$$m^a \ddot{q}_a + k^a q_a = 0 \tag{21}$$

The eigenvector solution to Eq. (21) yields the modal substitution transformation as

$$q_a = \begin{Bmatrix} \bar{q} \\ \hat{q} \end{Bmatrix}_a = \phi_a \xi_a = \begin{bmatrix} \bar{\phi} \\ \hat{\phi} \end{bmatrix}_a \xi_a \tag{22}$$

Equation (22) and the corresponding eigenvalues now represent the modes of Component a, and include approximate dynamic effects from Component b. Component modes determined in this manner are modified to resemble system modes. Using this type of component modes will improve the accuracy of the system results. The component mode substitution shown by Eq. (22) may also be determined without the reduced mass and stiffness matrix of Component b. All subsequent component mode substitutions in the method may be made using modes that were obtained either with or without interface loading. When interface loading is not used, Eqs. (10) and (13) are deleted.

The above procedure illustrates the concept of stiffness and mass-interface loading of Component a. For the general case including any number of components, Component b represents all other components of the system coupled together.

System Modes Using Constrained Component Branch Modes

In the component mode substitution method using constrained component branch modes, one component is selected as the main body and all other components are designated as branch components. (A branch component is a component whose motion will be defined relative to another component.) The coordinates of each branch are then transformed successively into main-body coordinates. The main-body component is considered to be free-free, and therefore, free-free component modes (including rigid-body modes) are used. Component branch modes are determined with the interface between the branch and the main body fixed.

The method of coupling the system is to use a coordinate transformation that will transform the coordinates of the uncoupled components into those of a coupled system, in which the component modes are referenced as generalized coordinates. This transformation is formed by combining together a series of intermediate transformations.

For component b, a new set of generalized coordinates is introduced. According to Bajan and Feng,[6] the noninterface displacements of Component b may be expressed as a superposition of constraint-mode and fixed-constraint-mode displacements. The constraint-modes are defined as a set of elastic and rigid-body displacements that occur at the noninterface coordinates $\hat{q}$ due to successive unit displacements at the interface coordinates $\bar{q}$. This definition was used to define $\hat{q}_{cb}$ in Eq. (4). Hence, the columns of matrix T_{cb} [see Eq. (4)] are the constraint modes. The fixed-constraint-mode displacements $\hat{q}_n$ define the elastic displacements of the noninterface coordinates $\hat{q}$ relative to the fixed interface coordinates $\bar{q}$. These relationships can be used to express the noninterface displacements of Component b as

$$\hat{q}_b = \hat{q}_{cb} + \hat{q}_{nb} \tag{23}$$

Then, from Eq. (23) and (4), the constraint-mode transformation $T_1{}^c$ can be written as

$$\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \bar{q}_b \\ \hat{q}_b \end{Bmatrix} = T_1{}^c \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \bar{q}_b \\ \hat{q}_{nb} \end{Bmatrix} \tag{24}$$

where

$$T_1{}^c = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & T_{cb} & I \end{bmatrix} \tag{25}$$

Components a and b are coupled together by the relationship shown in Eq. (15). From Eq. (15), the Component b-to-Component a coupling transformation $T_2{}^c$, can be formed as

$$\begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \bar{q}_b \\ \hat{q}_{nb} \end{Bmatrix} = T_2{}^c \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \\ q_{nb} \end{Bmatrix} \tag{26}$$

where

$$T_2{}^c = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{27}$$

Equations (24) and (26) can then be combined into a more concise transformation called $T_3{}^c$:

$$\begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \bar{q}_b \\ \hat{q}_b \end{Bmatrix} = T_3{}^c \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \hat{q}_{nb} \end{Bmatrix}, \tag{28}$$

where

$$T_3{}^c = T_1{}^c T_2{}^c \tag{29}$$

Equation (28) defines the complete motion of coupled Components a and b in terms of the coordinates of Component a and the fixed-constraint-mode coordinates of Component b.

In the general case where there are more than two components, further interface partitioning is done and Eqs. (24)–(29) will be used successively, starting from the most remote branch until all components are coupled.

The fixed-constraint-mode displacements of Component b, $\hat{q}_{nb}$, may be replaced via a component mode substitution using fixed interface modes. The fixed-interface equations of motion for undamped Component b can be written as

$$m_{22b}\ddot{\hat{q}}_b + k_{22b}\hat{q}_b = 0 \tag{30}$$

The eigenvector solution to Eq. (30) yields the following modal substitution for the fixed interface of Component b:

$$\begin{aligned} \hat{q}_b \text{ (for } \bar{q}_b = 0 \text{ and } \ddot{\bar{q}}_b = 0) \\ = \hat{q}_{nb} = \hat{\phi}_b{}^c \xi_b{}^c \end{aligned} \tag{31}$$

Component mode substitutions are now available for both Component a and Component b. From Eqs. (22) and (31), the combined component mode substitution transformation, $T_4{}^c$, can be written as

$$\begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \hat{q}_{nb} \end{Bmatrix} = T_4{}^c \begin{Bmatrix} \xi_a \\ \xi_b{}^c \end{Bmatrix} \tag{32}$$

where

$$T_4{}^c = \begin{bmatrix} \bar{\phi}_a & 0 \\ \hat{\phi}_a & 0 \\ 0 & \hat{\phi}_b{}^c \end{bmatrix} \tag{33}$$

For the general case in which there are more than two components, Eqs. (32) and (33) are expanded to include constrained component modal substitutions for all branches.

The system coordinate transformation, $T_5{}^c$, which will transform the uncoupled discrete coordinates of Components a and b to those of a coupled system in which component modes are described as generalized coordinates, can now be formed. Equations (28) and (32) can be combined to form

$$\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \bar{q}_b \\ \hat{q}_b \end{Bmatrix} = T_5{}^c \begin{Bmatrix} \xi_a \\ \xi_b{}^c \end{Bmatrix} \tag{34}$$

where

$$T_5{}^c = T_3{}^c T_4{}^c = \begin{bmatrix} \bar{\phi}_a & 0 \\ \hat{\phi}_a & 0 \\ \bar{\phi}_a & 0 \\ T_{cb}\bar{\phi}_a & \hat{\phi}_b{}^c \end{bmatrix} \tag{35}$$

From the system coordinate transformation defined by Eq. (34), the system modes can be determined. The total potential energy for the uncoupled Components a and b can be written from Eq. (14) in the form

$$(PE_T) = \tfrac{1}{2}[q_a{}^T q_b{}^T] \begin{bmatrix} k_a & 0 \\ 0 & k_b \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \end{Bmatrix} \tag{36}$$

Using Eq. (36) and (34), the potential energy for the coupled system that contains only component modal coordinates can be formed as

$$(PE)_{ab} = \tfrac{1}{2}[\xi_a{}^T \xi_b{}^{cT}] K^c \begin{Bmatrix} \xi_a \\ \xi_b{}^c \end{Bmatrix} \tag{37}$$

where

$$K^c = T_5{}^{cT} \begin{bmatrix} k_a & 0 \\ 0 & k_b \end{bmatrix} T_5{}^c \tag{38}$$

By taking the time derivative of Eq. (34), the system modal mass matrix may be formed from the kinetic energy, in a similar manner, as

$$(KE)_{ab} = \tfrac{1}{2}[\dot{\xi}_a{}^T \dot{\xi}_b{}^{cT}] M^c \begin{Bmatrix} \dot{\xi}_a \\ \dot{\xi}_b{}^c \end{Bmatrix} \tag{39}$$

where

$$M^c = T_5{}^{cT} \begin{bmatrix} m_a & 0 \\ 0 & m_b \end{bmatrix} T_5{}^c \tag{40}$$

The homogeneous equations of undamped motion for the system can be written, using only component modal coordinates from Eqs. (37) and (39); as

$$M^c \begin{Bmatrix} \ddot{\xi}_a \\ \ddot{\xi}_b{}^c \end{Bmatrix} + K^c \begin{Bmatrix} \xi_a \\ \xi_b{}^c \end{Bmatrix} = 0 \tag{41}$$

The eigenvector solution to Eq. (41) yields the following modal substitution transformation for the system;

$$\begin{Bmatrix} \xi_a \\ \xi_b{}^c \end{Bmatrix} = \phi_m{}^c \xi_m{}^c \tag{42}$$

The displacements of Components a and b are found by substituting Eq. (42) into Eq. (34) and result in

$$\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \phi_D{}^c \xi_m{}^c \tag{43}$$

where

$$\phi_D{}^c = T_5{}^c \phi_m{}^c \tag{44}$$

Equation (44) represents the component displacements of the system modes.

It is interesting to note that combining Eq. (28) and (36) results in

$$(PE)_{ab} = \tfrac{1}{2}[\bar{q}_a{}^T \hat{q}_a{}^T \hat{q}_{n_b}{}^T]\, k_{ab}{}^c \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \hat{q}_{n_b} \end{Bmatrix} \tag{45}$$

where

$$k_{ab}{}^c = T_3{}^{c^T} \begin{bmatrix} k_a & 0 \\ 0 & k_b \end{bmatrix} T_3{}^c \tag{46}$$

Expanding Eq. (46) results in

$$k_{ab}{}^c = \left[\begin{array}{cc|c} k_{11_a} + \bar{k}_b & k_{12_a} & 0 \\ k_{21_a} & k_{22_a} & 0 \\ \hline 0 & 0 & k_{22_b} \end{array}\right] \tag{47}$$

An inspection of Eq. (47) shows that the reduced stiffness properties of Component b, previously defined by Eq. (10), have been added to the upper left partition, and that the constrained stiffness matrix for Component b occurs in the lower right partition. No stiffness-coupling terms occur between Components a and b; instead, coupling between Components a and b occurs only in the corresponding system mass matrix. Hence, this part of the method could be called "generalized inertia coupling."

It should also be noted that the component modes substituted in Eq. (33) for Component a need not be the interface-loaded mass and stiffness modes of Eq. (22). Any set of free-free component modes may be used for Component a. However, if the interface-loaded mass and stiffness modes defined by Eq. (22) are used, Eq. (37) will be expanded into the form

$$(PE)_{ab} = \tfrac{1}{2}[\xi_a{}^T \xi_b{}^{c^T}] \begin{bmatrix} K^a & 0 \\ 0 & K_b{}^c \end{bmatrix} \begin{Bmatrix} \xi_a \\ \xi_b{}^c \end{Bmatrix} \tag{48}$$

where

$$K^a = \phi_a{}^T k^a \phi_a = \left[\diagdown \beta_i \omega_i{}^2 \diagdown\right]^a \tag{49}$$

and

$$K_b{}^c = \hat{\phi}_b{}^{c^T} k_{22_b} \hat{\phi}_b{}^c = \left[\diagdown \beta_i \omega_i{}^2 \diagdown\right]_b \tag{50}$$

The system modal stiffness matrix K^c, which was defined by Eq. (38) will become diagonal due to orthogonal mode properties.

System Modes Using Free-Free Component Modes

In this part of the component mode substitution method, free-free component modes (including rigid-body modes) are used for all components.

Component b is coupled to Component a using Eq. (15) to form the coupling transformation, T_2, which can be written as

$$\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \bar{q}_b \\ \hat{q}_b \end{Bmatrix} = T_2 \begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \hat{q}_b \end{Bmatrix} \tag{51}$$

where

$$T_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{52}$$

When there are more than two components, further interface partitioning is done. Equation (51) is expanded to couple all components together.

The free-free modes of Component a which contain the effects of the reduced-stiffness and mass matrices of Component b, are shown in Eq. (22). Similarly, the free-free modes of Component b are determined so that they contain the effects of the reduced-stiffness and mass matrices of Component a:

$$q_b = \begin{Bmatrix} \bar{q} \\ \hat{q} \end{Bmatrix}_b = \phi_b \xi_b = \begin{bmatrix} \bar{\phi} \\ \hat{\phi} \end{bmatrix}_b \xi_b \tag{53}$$

The component b modal substitution transformation written from the lower partition of Eq. (53) becomes

$$\hat{q}_b = \hat{\phi}_b \xi_b \tag{54}$$

Component mode substitutions are available for both components. The combined component mode substitution transformation T_4 for Components a and b is formed from Eq. (22) and (54) as

$$\begin{Bmatrix} \bar{q}_a \\ \hat{q}_a \\ \hat{q}_b \end{Bmatrix} = T_4 \begin{Bmatrix} \xi_a \\ \xi_b \end{Bmatrix} \tag{55}$$

where

$$T_4 = \begin{bmatrix} \bar{\phi}_a & 0 \\ \hat{\phi}_a & 0 \\ 0 & \hat{\phi}_b \end{bmatrix} \tag{56}$$

When there are more than two components, Eqs. (55) and (56) are expanded to include additional component modal substitutions for all the components in the system.

The system coordinate transformation T_5 which will transform the uncoupled coordinates of Components a and b to those of a coupled system in which the component modes are used as the generalized coordinates can be formed. Equations (51) and (55) can be combined to form

$$\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \begin{bmatrix} \bar{q}_a \\ \hat{q}_a \\ \bar{q}_b \\ \hat{q}_b \end{bmatrix} = T_5 \begin{Bmatrix} \xi_a \\ \xi_b \end{Bmatrix} \tag{57}$$

where

$$T_5 = T_2 T_4 = \begin{bmatrix} \bar{\phi}_a & 0 \\ \hat{\phi}_a & 0 \\ \bar{\phi}_a & 0 \\ 0 & \hat{\phi}_b \end{bmatrix} \tag{58}$$

Using the system coordinate transformation defined by Eq. (57), the system modes can be found. The total potential energy for the system can be formed by combining Eqs. (57) and (36)

$$(PE)_{ab} = \tfrac{1}{2}[\xi_a{}^T \xi_b{}^T]\, K \begin{Bmatrix} \xi_a \\ \xi_b \end{Bmatrix} \tag{59}$$

where

$$K = T_5{}^T \begin{bmatrix} k_a & 0 \\ 0 & k_b \end{bmatrix} T_5 \tag{60}$$

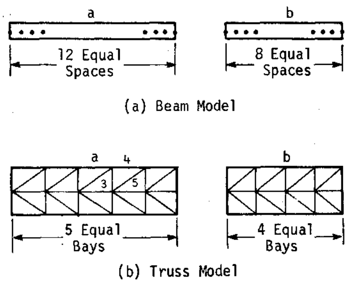


Fig. 2 Typical models.

By taking the time derivative of Eq. (57), the system modal mass matrix may be formed from the kinetic energy in a similar manner as

$$(KE)_{ab} = \tfrac{1}{2}[\dot{\xi}_a{}^T \dot{\xi}_b{}^T] M \begin{Bmatrix} \dot{\xi}_a \\ \dot{\xi}_b \end{Bmatrix} \tag{61}$$

where

$$M = T_5{}^T \begin{bmatrix} m_a & 0 \\ 0 & m_b \end{bmatrix} T_5 \tag{62}$$

The homogeneous equations of undamped motion for the system can be written, using only free-free component modal coordinates from Eq. (59) and (61), as

$$M \begin{Bmatrix} \ddot{\xi}_a \\ \ddot{\xi}_b \end{Bmatrix} + K \begin{Bmatrix} \xi_a \\ \xi_b \end{Bmatrix} = 0 \tag{63}$$

The eigenvector solution to Eq. (63) yields the following modal substitution transformation:

$$\begin{Bmatrix} \xi_a \\ \xi_b \end{Bmatrix} = \phi_m \xi_m \tag{64}$$

The displacements of Components a and b are found by substituting Eq. (64) into Eq. (57) and results in

$$\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \phi_D \xi_m \tag{65}$$

where

$$\phi_D = T_5 \phi_m \tag{66}$$

Equation (66) represents the component displacements of the system modes.

The free-free Component b modal substitution transformation defined by Eq. (54) may be replaced by another method. The interface coordinates $\bar{q}_b$ in Eq. (2) can be reduced, leaving only noninterface coordinates $\hat{q}_b$ The corresponding reducing transformation and its time derivative can be substituted into Eqs. (8) and (11), respectively. The resulting potential and kinetic energies will contain $\hat{q}_b$ and $\dot{\hat{q}}_b$ only as generalized coordinates. The corresponding eigenvector solution can then be used as the component mode substitution to replace Eq. (54). This alternative method is useful for reducing the number of component coordinates where there are too many to be used in available eigenvalue-eigenvector computer programs.

Small changes can be made to the mass and stiffness properties of components without having to recalculate the interface-loaded component modes. The component mass and stiffness matrices used in Eqs. (38, 40, 60, and 62) contain the revised mass and stiffness matrices.

The number of generalized coordinates used to determine the system modes in Eqs. (41) and (63) is controlled by truncating the number of component modes used in Eqs. (33) and (56) The component modal substitutions that are used need not be the interface-loaded mass and stiffness modes in Eq. (22). For the same number of component modes, more accurate results are obtained using interface-loaded modes. This effect is shown in the numerical examples.

For some applications, it may be useful to determine the component modes using only translational coordinates. If rotational coordinates exist, they will be mixed with component modal coordinates in Eqs. (41) and (63). The rotational coordinates can subsequently be eliminated, in a manner similar to that shown in Eqs. (2–7), by determining a reduction transformation that will use the modal stiffness matrices K^c or K. This reducing transformation is then used in Eqs. (37, 39, 59, and 61) to leave only component mode coordinates.

System constraints may exist, and are assumed to be included in the basic mass and stiffness matrices of the components. No modifications to the method are necessary.

Two approaches may be used to handle a large number of components. One is to couple all components simultaneously, as has been indicated in the preceding derivations. The second, which is useful if restrictions on the size of the computer program prevent the simultaneous approach from being used, consists of cascading the components successively. First, two components are coupled together. Next, three of the components are coupled together, and the first two coupled components are treated as one; the previously calculated system component modes are then used as the component mode substitutions. More than two interconnected components may be treated using this method. This will require additional connections to be made if constrained branch modes are used. A transformation similar to Eq. (16) may be formed and inserted as an additional transformation either before Eq. (24) or after Eq. (28). The use of free-free component modes requires no further explanation.

An interesting aspect of using the above methods to handle many components is that a combination of constrained and free-free component modes can be intermixed simultaneously. This, however, must be done carefully, and requires a complete understanding of the method presented.

Numerical Results

Two examples have been considered: a beam and a planar truss. The beam is shown in Fig. 2(a) and consists of two beam components that have uniform stiffness and mass properties. One component consists of 13, and the other, of nine, equally-spaced collocation points, each having two degrees of freedom.

The planar truss, shown in Fig. 2(b), consists of two truss components having uniform bay sections. All members have a constant area and uniform mass properties. One component consists of five equal bays and has a total of 18 joints The other component consists of four equal bays and has a total of 15 joints. Each joint has two degrees of free-

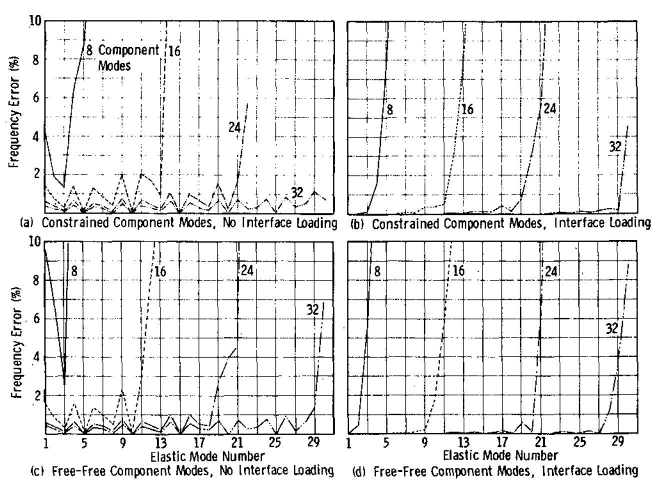


Fig. 3 Frequency error vs elastic mode number for beam model.

dom. The relative lengths of the members are indicated in Fig. 2(b).

The system modes for both examples were determined using four types of component modes: 1) Constrained component branch modes with no interface loading, 2) Constrained component branch modes with interface loading, 3) Free-free component modes with no interface loading; 4) Free-free component modes with interface loading.

For the beam, 8, 16, 24, and 32 component modes were used; for the truss, 12, 20, 28, and 36 component modes were used. A frequency cutoff criterion was used to select the component modes: i.e., all component modes whose frequencies were less than or equal to the cutoff frequency were used.

Figures 3(a) and 3(b) present the system frequency error for the beam using constrained component branch modes. Forty-two component modes were available. Figure 3(a) is for the no-interface-loading condition; good results were obtained except for the 8-component-mode case. Figure 3(b), which is for the interface-loading condition, shows that significantly better accuracy was obtained. Note that the 8-component-mode case gave very accurate low modes.

Figures 3(c) and 3(d) present the system frequency error for the beam using free-free component modes Forty-four component modes were available. Figure 3(c) shows the results for the no-interface-loading condition, which produced good results except for the 8-component-mode case. Figure 3(d) which is for the interface-loading condition, again shows that a significant accuracy improvement was obtained for the low modes. In general, the higher modes obtained by using free-free component modes were not as accurate as those obtained by using constrained component branch modes.

Figures 4(a) and 4(b) present the system frequency error for the truss using constrained component branch modes. Sixty component modes were available. Figure 4(a) shows that for the no-interface-loading condition, good results were obtained except for the 12-component-mode case. Figure 4(b) is for the interface-loading condition; significant accuracy improvement was obtained for the lower modes. The 12-mode case gave very accurate low modes.

Figures 4(c) and 4(d) present the system frequency error for the truss using free-free component modes. Sixty-six component modes were available. Figure 4(c) is for the no-interface-loading condition and shows that good results were obtained using 28 and 36 component modes. Figure 4(d) is for the interface-loading condition; very significant accuracy improvement was obtained for the lower modes. The results obtained using constrained component branch modes were, generally, more accurate, especially in the higher modes, than the results obtained using free-free component modes.

Interface loading on the component modes primarily improved the low modes and was unpredictable for the high

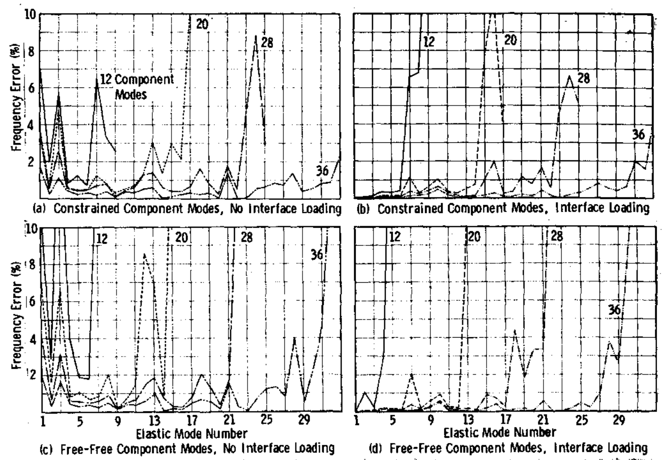


Fig. 4 Frequency error vs elastic mode number for truss model.

modes. The improvement in accuracy obtained by using interface-loaded-component modes decreased as the number of component modes increased.

References

[1] Hunn, B. A., "A Method of Calculating the Normal Modes of an Aircraft," *Quarterly Journal of Mechanics*, Vol. 8, Pt. 1, 1955, pp. 38–58.

[2] Gladwell, G. M. L., "Branch Modes Analysis of Vibrating Systems," *Journal of Sound Vibration*, Vol. I, 1964, pp. 41–59.

[3] Hurty, W. C., "Vibrations of Structural Systems by Component Mode Synthesis," *Proceedings of American Society of Civil Engineers*, Vol. 85, No. EM4, Aug. 1960, pp. 51–69.

[4] Hurty, W. C., "Dynamic Analysis of Structural Systems by Component Mode Synthesis," TR 32-530, Jet Propulsion Lab., January 15, 1964, Pasadena, Calif.

[5] Hurty, W. C., "Dynamic Analysis of Structural Systems Using Component Modes," *AIAA Journal*, Vol. 3, No. 4, April 1965, pp. 678–685.

[6] Bajan, R. L. and Feng, C. C., "Free Vibration Analysis by the Modal Substitution Method," AAS Paper 68-8-1, July 1968.

[7] Craig, R. R., and Bampton, M. C. C., "Coupling of Substructures for Dynamic Analysis," *AIAA Journal*, Vol. 6, No. 7, July 1968, pp. 1313–1319.

[8] Goldman, R. L., "Vibration Analysis by Dynamic Partitioning," *AIAA Journal*, Vol. 7, No. 6, June 1969, pp. 1152–1154.

[9] Hou, S. N., "Review of Modal Synthesis Techniques and A New Approach," *The Shock and Vibration Bulletin*, Bulletin 40, Pt. 4, Naval Research Lab., Washington, D.C., Dec. 1969, pp. 25–39.

[10] Przemieniecki, J. S., "Theory of Matrix Structural Analysis," McGraw-Hill, New York, 1968, pp. 122–128 and pp. 291–292.